Algebras of Measurements: the logical structure of Quantum Mechanics *

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July 2005, modified September 2005

Abstract

In Quantum Physics, a measurement is represented by a projection on some closed subspace of a Hilbert space. We study algebras of operators that abstract from the algebra of projections on closed subspaces of a Hilbert space. The properties of such operators are justified on epistemological grounds. Commutation of measurements is a central topic of interest. Classical logical systems may be viewed as measurement algebras in which all measurements commute. Keywords: Quantum measurements, Measurement algebras, Quantum Logic. PACS: 02.10.-v.

*This work was partially supported by the Jean and Helene Alfassa fund for research in Artificial Intelligence, by the Israel Science Foundation grant 183/03 on "Quantum and other cumulative logics" and by EPSRC Visiting Fellowship GR/T 24562 on "Quantum Logic"

1 Introduction

We define a new class of abstract structures for which we coin the term *algebras of measurements*, M-algebras for short. In this expression the term *algebra* must be understood in its most general acception. The elements of those structures are *not* linear operators on some vector space. They are intended to capture the logic of physical measurements and in particular of quantum measurements. From the physicist's point of view, it provides a framework, devoid of real physics and numbers, in which both classical and quantum mechanics can be described and the difference between them put in evidence. Classical physics is, there, a special, very limited, almost trivial, case of quantum physics. From the logician's point of view, it provides a generalized view of (non-monotonic) logic in which classical, i.e. monotonic, logic is a special, very limited, almost trivial case.

This work takes its inspiration from the pioneering work of G. Birkhoff and J. von Neumann [2]. They proposed the view that *experimental propositions* are closed subspaces of a Hilbert space and measurements are projections on such closed subspaces. Strangely enough, they presented only a very preliminary analysis of the properties of such projections and this topic seems to have been almost ignored since. It had to wait for the thoroughly new point of view proposed by Engesser and Gabbay in [4]. In this paper, we take the following views:

- the Hilbertian formalization of Quantum Physics has been so extremely successful for the reason that the algebra of projections in Hilbert spaces possesses the properties that are epistemologically necessary to deal with measurements that change the state of the system measured. We shall therefore provide epistemological justifications to the properties possessed by the Hilbertian formalism,

- not all properties of the Hilbertian formalism are epistemologically justified with the same force. Some (logical) aspects of Quantics may, probably, be studied, with advantage, in a weaker context,

- the formalism of Measurement algebras suggests some, logically based, principles akin to superselection rules.

Let us, first, develop the analogy between measurements and propositions. A physical measurement, e.g., measuring the temperature of a gas to be $138\,^\circ K$, asserts that the proposition *the temperature of this gas is* $138\,^\circ K$ holds true. A measurement, in a sense, asserts the truth of a proposition. This is the fundamental analogy between physics and logic: making a measurement is similar to asserting a certain kind of proposition. The example above has been taken from classical physics. Consider now measuring the spin of a particle along the z-axis to be 1/2. This measurement is akin to asserting the truth of the proposition *the spin along the z-axis is* 1/2. But, here, the assertion of the proposition, i.e., the measurement, changes the state of the system. The assertion holds in the state resulting from the measurement, but did not necessarily hold in the state of the system before the measurement was performed. In fact it held in this previous state if and only if the measurement left the state unchanged. Inspired by the analogy between measurements and propositions we set ourselves to study the logic of propositions that not only *hold* at states, i.e., models, but also operate on them, transforming the state in which they are evaluated into another one. A proposition holds in some state if and only if this state is a fixpoint for the proposition.

Section 2 will summarize in a most succinct and formal way the definition of Algebras of Measurements (M-algebras), by presenting a list of properties. It should be used as an overview and memento only. The following sections will explain the properties, present motivation and explanation, and then prove basic properties of M-algebras.

2 M-algebras

The structures we are concerned with deal with a set X and functions from X to X. We shall denote the composition of functions by $\circ$ and composition has to be understood from left to right: for any $x \in X$, $(\alpha \circ \beta)(x) = \beta(\alpha(x))$. If $\alpha : X \longrightarrow X$, we shall denote by $FP(\alpha)$ the set of all fixpoints of α: $FP(\alpha) \stackrel{\text{def}}{=} \{x \in X \mid \alpha(x) = x\}$.

Definition 1 *An M-algebra is a pair $\langle X, M \rangle$ in which X is a non-empty set and M is a set of functions from X to X, that satisfies the six properties described below.*

- **Illegitimate** $\exists\, 0 \in X$ such that $\forall \alpha \in M$, $0 \in FP(\alpha)$, i.e, $\alpha(0) = 0$.
- **Idempotence** $\forall \alpha \in M$, $\alpha \circ \alpha = \alpha$, i.e., for any $x \in X$, $\alpha(\alpha(x)) = \alpha(x)$.

The next property requires a preliminary definition.

Definition 2 *For any $\alpha, \beta : X \longrightarrow X$, we shall say that α* preserves *β if and only if α preserves $FP(\beta)$, i.e., if $\alpha(FP(\beta)) \subseteq FP(\beta)$, i.e., $\forall x \in X,\ \beta(x) = x \Rightarrow \beta(\alpha(x)) = \alpha(x)$.*

- **Composition** $\forall \alpha, \beta \in M$, if α preserves β, then $\beta \circ \alpha \in M$.
- **Interference** $\forall x \in X,\ \forall \alpha, \beta \in M,$ if $x \in FP(\alpha)$, i.e., $\alpha(x) = x$, and $(\beta \circ \alpha)(x) \in FP(\beta)$, i.e., $\beta(\alpha(\beta(x))) = \alpha(\beta(x))$, then $\beta(x) \in FP(\alpha)$, i.e., $\alpha(\beta(x)) = \beta(x)$.
- **Cumulativity** $\forall x \in X, \forall \alpha, \beta \in M$, if $\alpha(x) \in FP(\beta)$, i.e., $\beta(\alpha(x)) = \alpha(x)$ and $\beta(x) \in FP(\alpha)$, i.e., $\alpha(\beta(x)) = \beta(x)$, then $\alpha(x) = \beta(x)$.

  The next property requires some notation. For any $\alpha : X \longrightarrow X$, we shall denote by $Z(\alpha)$ the set of zeros of α: $Z(\alpha) \stackrel{\text{def}}{=} \{x \in X \mid \alpha(x) = 0\}$.
- **Negation** $\forall \alpha \in M, \exists(\neg\alpha) \in M$, such that $FP(\neg\alpha) = Z(\alpha)$, and $Z(\neg\alpha) = FP(\alpha)$, i.e., $\forall x \in X,\ \alpha(x) = 0$ iff $(\neg\alpha)(x) = x$ and $\forall x \in X,\ \alpha(x) = x$ iff $(\neg\alpha)(x) = 0$.

Two additional properties will be considered in Section 9.1.

Definition 3 *An M-algebra is* separable *if it satisfies the following:*

Separability *For any $x, y \in X - \{0\}$, if $x \neq y$ then $\exists \alpha \in M$ such that $\alpha(x) = x$ and $\alpha(y) \neq y$.*

Definition 4 *An M-algebra is* strongly separable *if it satisfies the following:*

Strong Separability *For any $x \in X - \{0\}$, there exists a measurement $e_x \in M$ such that $FP(e_x) = \{0, x\}$.*

3 Motivation and Justification

In this section, we shall leisurely explain each one of the properties described in Section 2. Our explanation of each property will include three parts:

- an epistemological explanation whose purpose is to explain why the property is natural or even required when one thinks of measurements,

- an explanation of why the property holds in the algebra $\langle H, L \rangle$ where H is a Hilbert space and L the set of all projections onto closed subspaces of H,

- an explanation of the logical meaning of the property, based on the identification of measurements with propositions.

3.1 States

We shall reserve the term *state* for the elements of X. In physical terms, the set X is the set of all possible states of a system. When we say *state* we mean a state as fully determined as is physically possible: e.g., in classical mechanics, a set of $6n$ values if we consider n particles (three values for position and three values for momentum), or what is generally termed, in Quantum Physics a *pure state*.

In the Hilbertian description of Quantum Physics, a (pure) state is a one-dimensional subspace, i.e., a ray, in some Hilbert space. The illegitimate state, 0 is the zero-dimensional subspace.

A logician can give the term "states" two different acceptions. It sometimes means a full definition of a possible state of the world, i.e., what is true and what is false. This is a model, a maximal consistent theory. Sometimes, it means a set of propositions known to be true. In this sense, it is any consistent theory, not necessarily maximal. We shall see that **Illegitimate** forces us to consider the inconsistent theory as a state.

3.2 Measurements

The elements of M represent measurements on the physical system whose possible states are those of X. In Classical Physics one may assume that a measurement leaves the measured system unchanged. It is a hallmark of Quantum Physics that this assumption cannot be held true anymore. In Quantum Physics, measurements, in general, change the state of the system. This is the phenomenon called *collapse of the wave function*. Therefore we model measurements by transformations on the set of states. Clearly not any transformation can be called a measurement. A measurement changes the system in some minimal way. A transformation that brings about a wild change in the system cannot be considered to be a measurement. Many of the properties presented above and discussed below explicit this requirement.

A word of caution is necessary here before we proceed. When we speak about measurement we do not mean some declaration of intentions such as *measuring the position of a particle*, we mean the action of measuring some physical quantity *and finding a specific value*, such as *finding the particle at the origin of the system of coordinates*. Measuring $0.3\,^{\circ}K$ and measuring $1000\,^{\circ}K$ are not two different possible results for the same measurement, they are two different measurements.

In the Hilbertian description of Quantum Physics measurable quantities are represented by Hermitian operators. Measurements in our sense are represented by a pair $\langle A, \lambda\rangle$ where A is a Hermitian operator and λ an eigenvalue of A. The effect of measuring $\langle A, \lambda\rangle$ in state x is to project x onto the eigensubspace of A for eigenvalue λ. A measurement α is therefore a projection on a closed subspace of a Hilbert space. The set $FP(\alpha)$ is the closed subspace on which α projects. Those projections onto eigensubspaces are the measurements we try to identify. Our goal is to identify the algebraic properties of such projections that make them suitable to represent physical measurements in Quantum Physics.

From a classical logician's point of view, a measurement is a proposition. A proposition α acts on a state, i.e., a theory T by sending it to the theory that results from adding α to T and then closing under logical consequence. One sees that, from this point of view, if T is maximal then $\alpha(T)$ is either T (iff α is in T) or the inconsistent theory. We see here that a proposition (measurement) holds in some model (state) if and only if the model is a fixpoint of the proposition.

This is the interpretation that we shall take along with us: a measurement α *holds* at some state x, or, equivalently x satisfies α, if and only if $x \in FP(\alpha)$.

3.3 Illegitimate

Illegitimate is mainly a technical requirement. The sequel will show why it is handy. The illegitimate state 0 is a state that is physically impossible. Physicists, in general, do not consider this state explicitly, we shall. From the epistemological point of view, we just require that amongst all the possible states of the system we include a state, denoted 0 that represents physical impossibility. There is not much sense in measuring anything in the illegitimate state, therefore, it is natural to assume that no measurement α operating on the illegitimate state can change it into some legitimate state. This is the meaning of our requirement that 0 be a fixpoint of any measurement. In other terms, the state 0 satisfies every measurement, every measurement holds at 0.

In the Hilbertian description of Quantum Physics the zero vector plays the role of our 0. Indeed, since a projection is linear, it preserves the zero vector.

From a logician's point of view **Illegitimate** requires us to include the inconsistent theory in X. Clearly, the result of adding any proposition to the inconsistent theory leaves us with the inconsistent theory.

3.4 Zeros

We have described in Section 3.2 the interpretation we give to the fact that a state x is a fixpoint of a measurement α. We want to give a similarly central meaning to the fact that a state x is a zero of a measurement α: $x \in Z(\alpha)$, i.e., $\alpha(x) = 0$. If measuring α sends x to the illegitimate state, measuring α is physically impossible at x. This should be understood as meaning that the state x has some definite value different from the one specified by α.

If, at x, the spin is $1/2$ along the z-axis, then measuring along the z-axis a spin of $-1/2$ is physically impossible and therefore the measurement of $-1/2$ sends the state x to the illegitimate state 0. The status of the measurement that measures $-1/2$ along the *x-axis* is completely different: this measurement does not send x to 0, but to some legitimate state in which the spin along the x-axis is $-1/2$.

It is natural to say that a measurement α has a definite value at x iff x is either a fixpoint or a zero of α. We shall define: $Def(\alpha) \stackrel{\text{def}}{=} FP(\alpha) \cup Z(\alpha)$. If $x \in Def(\alpha)$, α has a definite value at x: either it holds at x or it is impossible at x. If $x \notin Def(\alpha)$, $\alpha(x)$ is some state different from x and different from 0.

In the Hilbertian presentation of Quantum Physics, the zeros of a measurement α are the vectors orthogonal to the set of fixpoints of α.

3.5 Idempotence

Idempotence is extremely meaningful. It is an epistemologically fundamental property of measurements that they are idempotent: if α is a measurement and x a state, then $\alpha(\alpha(x)) = \alpha(x)$, i.e., measuring the same value twice in a row is exactly like measuring it once. Note that, by **Illegitimate**, if $x \in Def(\alpha)$, then $\alpha(\alpha(x)) = \alpha(x)$. The import of **Idempotence** concerns states that are not in $Def(\alpha)$.

It seems very difficult to imagine a scientific theory in which measurements are not idempotent: it would be impossible to check directly that a system is indeed in the state we expect it to be in without changing it. Idempotence is one of the conditions that ensure that measurements change states only minimally. This principle seems to be a fundamental principle of all science, having to do with the reproducibility of experiments. If there was a physical system and a measurement that, if performed twice in a row gave different results, then such a measurement would be, in principle, irreproducible.

In the Hilbertian description of Quantum Physics measurements are modeled by projections onto eigensubspaces. Any projection is idempotent. But it is enlightening to reflect on the phenomenology of this idempotence. For an electron whose spin is positive along the z-axis (state x_0), measuring a negative spin along the x-axis is feasible, i.e., does not send the system into the illegitimate state, but sends the system into a state (x_1) different from the original one, x_0. Nevertheless, a consequence of the collapse of the wave function is that, after measuring a negative spin along the x-axis, the spin is indeed negative along the x-axis and therefore a new measurement of a negative spin along the x-axis leaves the state x_1 of our electron unchanged, whereas measuring a positive spin along the x-axis is now an unfeasible measurement and sends x_1 to the illegitimate state. Note that such a measurement of a positive spin along the x-axis in the original state x_0 brings us to a legitimate state x_3 different from x_0 and x_1. The idempotence of measurements, probably epistemologically necessary, provides some explanation of why projections in Hilbert spaces are a suitable model.

From the logician's point of view, idempotence corresponds to the fact that asserting the truth of a proposition is equivalent to asserting it twice. For any reasonable consequence operation $\mathcal{C}$, $\mathcal{C}(\mathcal{C}(T, a), a) = \mathcal{C}(T, a)$.

3.6 Preservation

The definition of *preservation* encapsulates the way in which different measurements can interfere. If α preserves $FP(\beta)$, the set of states in which β holds, α never destroys the truth of proposition β: it never interferes badly with β.

3.7 Composition

Composition has physical significance. It is a global principle: it assumes a global property and concludes a global property. Measurements are mappings of X into itself, therefore we may consider the composition of two measurements. According to the principle of minimal change, we do not expect the composition of two measurements to be a measurement: two small changes may make a big change. But, if those two measurements do not interfere in any negative way with each other, we may consider their composition as small changes that do not add up to a big change. **Composition** requires that if, indeed, α preserves β, then the composite operation that consists of measuring β first, and then α does not add up to a big change and should be a bona fide measurement. Notice that we perform β first, whose result is (by **Idempotence**) a state that satisfies β, then we perform α, which does not destroy the result obtained by the first measurement β.

In the Hilbertian presentation of Quantum Physics, consider α, the projection on some closed subspace A and β, the projection on B. The measurement α preserves β iff the projection of the subspace B onto A is contained in the intersection $A \cap B$ of A and B. In such a case the composition $\beta \circ \alpha$ of the two projections, first on B and then on A is equivalent to the projection on the intersection $A \cap B$. It is therefore a projection on some closed subspace.

For the classical logician, measurements always preserve each other. If $a \in T$, then $a \in \mathcal{C}(T, b)$ for any proposition b. This is a consequence of the monotonicity of $\mathcal{C}$. **Composition** requires that the composition of any two measurements be a measurement. For the logician, $\beta \circ \alpha$ is the measurement $\beta \wedge \alpha$. **Composition** amounts to the assumption that M is closed under conjunction.

Technically, the role of **Composition** is to ensure that two commuting measurements' composition is a measurement. Equivalently, we could have, instead of **Composition**, required that for any pair $\alpha, \beta \in M$ such that $\alpha \circ \beta = \beta \circ \alpha$, their composition $\alpha \circ \beta$ be in M.

3.8 Interference

Interference has a deep physical meaning. It is a local principle, i.e., holds separately at each state x. It may be seen as a local logical version of Heisenberg's uncertainty principle. It considers a state x that satisfies α. Measuring β at x may leave α undisturbed (this is the conclusion), but, if β disturbs α, then no state at which both α and β hold can ever be attained by measuring α and β in succession. In other words, either such a state, satisfying both α and β is obtained immediately, or never.

We shall say that β disturbs α at x if $x \in FP(\alpha)$ but $\beta(x) \notin FP(\alpha)$. Note that β preserves α if and only if it disturbs α at no x. **Interference** says that if β disturbs α at x then α disturbs β at $\beta(x)$, and β disturbs α at $(\beta \circ \alpha)(x)$, and so on. We chose to name this property *Interference* since it deals with the local interference of two measurements: if they interfere once, they will continue interfering ad infinitum.

In the Hilbertian presentation of Quantum Physics, the principle of **Interference** is satisfied for the following reason. Consider a vector $x \in H$ and two closed subspaces of H: A and B. Assume x is in A. Let y be the projection of x onto B and z the projection of y onto A. Assume that z is in B. Since both x and z are in A, the vector $z - x$ is in A. Similarly, the vector $z - y$ is in B. But y is the projection of x onto B and therefore $y - x$ is orthogonal to B and in particular orthogonal to $z - y$. We have $(y - x) \cdot (z - y) = 0$, and

$$y \cdot z - y \cdot y - x \cdot z + x \cdot y = 0.$$

Since z is the projection of y onto A, the vector $z - y$ is orthogonal to A and we have $(z - x) \cdot (z - y) = 0$, and

$$z \cdot z - z \cdot y - x \cdot z + x \cdot y = 0.$$

By substracting the first equality from the second we get:

$$-z \cdot y - y \cdot z + y \cdot y + z \cdot z = (y - z) \cdot (y - z) = 0.$$

We conclude that $y = z$.

For the logician, it is always the case that $\beta(x) \in FP(\alpha)$ if $x \in FP(\alpha)$, as noticed in Section 3.7.

3.9 Cumulativity

Cumulativity is motivated by Logic. It does not seem to have been reflected upon by physicists. It parallels the cumulativity property that is central to nonmonotonic logic: see for example [5, 7, 6]. If the measurement of α at x causes β to hold (at $\alpha(x)$), and the measurement of β at x causes α to hold (at $\beta(x)$) then those two measurements have, locally (at x), the same effect. Indeed, they cannot be directly distinguished by testing α and β. **Cumulativity** says that they cannot be distinguished even indirectly.

In the Hilbertian formalism, if the projection, y, of x onto some closed subspace A is in B (closed subspace) then y is the projection of x onto the intersection $A \cap B$. If the projection z of x onto B is in A, z is the projection of x onto the intersection $B \cap A$ and therefore $y = z$. In fact, a stronger property than **Cumulativity** holds in Hilbert spaces. The following property, similar to the Loop property of [5], holds in Hilbert spaces: **L-Cumulativity** $\forall x \in X$, for any natural number n and for any sequence $\alpha_i \in M$, $i = 0, \ldots, n$ if, for any such i, $\alpha_i(x) \in FP(\alpha_{i+1})$, where $n+1$ is understood as 0, then, for any $0 \leq i, j \leq n$, $\alpha_i(x) = \alpha_j(x)$.

To see that this property holds in Hilbert spaces, consider the distance d_i between x and the closed subspace A_i on which α_i projects. The condition $\alpha_i(x) \in FP(\alpha_{i+1})$ implies that $d_{i+1} \leq d_i$. We have $d_0 \geq d_1 \geq \ldots \geq d_n \geq d_0$ and we conclude that all those distances are equal and therefore $\alpha_i(x) \in FP(\alpha_{i+1})$ implies that $\alpha_i(x) = \alpha_{i+1}(x)$. We do not know whether the stronger **L-Cumulativity** is meaningful for Quantum Physics, or simply an uninteresting consequence of the Hilbertian formalism.

For the logical point of view, one easily sees that any classical measurements satisfy **Cumulativity**, and even **L-Cumulativity**.

3.10 Negation

Negation also originates in Logic. It corresponds to the assumption that propositions are closed under negation. If α is a measurement, α tests whether a certain physical quantity has a specific value v. If such a test can be performed, it seems that a similar test could be performed to test the fact that the physical quantity of interest has some other specific value or does not have value v.

In the Hilbertian formalism, to any closed subspace corresponds its orthogonal subspace, also closed.

For the logician, **Negation** amounts to the closure of the set of (classical) measurements, i.e., formulas, under negation.

3.11 Separability

We remind the reader that none of the separability assumptions is included in the defining properties of an M-algebra. **Separability** asserts that if any two non-zero states x and y are different, there is a measurement that holds at x and not at y. Indeed, if all measurements that hold at x also hold at y it would not be possible to be sure that the system is in x and not in y. Compared to the previous requirements, **Separability** is of quite a different kind. It is some akin to a superselection principle, though presented in a dual way: a restriction on the set of states not on the set of observables.

Note that this implies that, in any non-trivial M-algebra (an M-algebra is trivial if $X = \{0\}$ and $M = \emptyset$), every state satisfies some measurement.

In the Hilbertian formalism, the projections on the one-dimensional subspaces defined by x and y respectively do the job.

For the logician, if T_1 and T_2 are two maximal consistent sets that are different, there is a formula α in $T_1 - T_2$. But, one may easily find (non-maximal) different theories T_1 and T_2 such that $T_1 \subset T_2$, contradicting **Separability**.

Strong Separability is a stronger requirement. Indeed, in a strongly separable M-algebra, for any non-zero, different, states x, y the measurement e_x holds at x and not at y.

The epistemological motivation for such a strong requirement is the following. One must be able to prepare a system in each of its states, i.e., each of the non-zero elements x of X. One this has been done, one should be able to check that indeed the system is in the state it is claimed to be, i.e., there should be a measurement that *measures* each non-zero state x: this measurement is e_x.

In the Hilbert space framework, every non-zero state is a one-dimensional subspace, therefore a closed subspace and a measurement. The same is true in classical mechanics: every state of a system can be characterized by one proposition stating all that is true about the system.

In the logical examples, we have seen above that considering all theories as states defines an algebra that is not even separable. Considering only maximal theories, on the contrary, provides for a strongly separable algebra, at least if there is only a finite set of atomic propositions, or if we admit infinite conjunctions and disjunctions.

4 Examples of M-algebras

In this section we shall formally define the two paradigmatical examples of M-algebras that have been described in Section 3: propositional calculus and Hilbert space.

4.1 Logical Examples

4.1.1 Propositional Calculus: a non-separable M-algebra and a separable one

We shall now formalize our treatment of Propositional Calculus as an M-algebra. In doing so, we shall present Propositional Calculus in the way advocated by Tarski and Gentzen. Let $\mathcal{L}$ be any language closed under a unary connective $\neg$ and a binary connective $\wedge$. Let $\mathcal{C}n$ be any consequence operation satisfying the following conditions (the conditions are satisfied by Propositional Calculus).

$$\textbf{Inclusion } \forall A \subseteq \mathcal{L},\ A \subseteq \mathcal{C}n(A),$$

$$\textbf{Monotonicity } \forall A, B \subseteq \mathcal{L}, A \subseteq B \ \Rightarrow\ \mathcal{C}n(A) \subseteq \mathcal{C}n(B),$$

$$\textbf{Idempotence } \forall A \subseteq \mathcal{L}, \mathcal{C}n(A) = \mathcal{C}n(\mathcal{C}n(A)),$$

$$\textbf{Negation } \forall A \subseteq \mathcal{L}, a \in \mathcal{L}, \mathcal{C}n(A, \neg a) = \mathcal{L} \ \Leftrightarrow\ a \in \mathcal{C}(A),$$

$$\textbf{Conjunction } \forall A \subseteq \mathcal{L}, a, b \in \mathcal{L},\ \mathcal{C}n(A, a, b) = \mathcal{C}n(A, a \wedge b).$$

Define a subset of $\mathcal{L}$ to be a *theory* iff it is closed under $\mathcal{C}n$: $T \subseteq \mathcal{L}$ is a theory iff $\mathcal{C}n(T) = T$. Let X be the set of all theories. Let M be the language $\mathcal{L}$. The action of a formula $\alpha \in \mathcal{L}$ on a theory T is defined by: $\alpha(T) = \mathcal{C}n(T \cup \{\alpha\})$. In such a structure α holds at T iff $\alpha \in T$. Let us check that such a structure satisfies all the defining properties of an M-algebra. We shall not mention the uses of Inclusion. The illegitimate state is the theory $\mathcal{L}$. **Idempotence** follows from the property of the same name. **Composition** follows from **Conjunction**: the composition $a \circ b$ is the measurement $a \wedge b$. Note that any pair of measurements commute. **Interference** is satisfied because $a \in T$ implies $a \in \mathcal{C}n(T, b)$. **Cumulativity** is satisfied because $b \in \mathcal{C}n(T, a)$ implies $\mathcal{C}n(T, a) = \mathcal{C}n(T, a, b)$ by **Monotonicity** and **Idempotence**. **Negation** holds by the property of the same name.

The M-algebra above does not satisfy **Separability** since there are theories T and S such that $T \subset S$ and every formula α satisfied by T is also satisfied by S. This M-algebra is *commutative*: any two measurements commute since: $\mathcal{C}n(\mathcal{C}(T,a),b) = \mathcal{C}n(\mathcal{C}(T,b),a)$.

If we consider the subset $Y \subset X$ consisting only of *maximal consistent* theories and the inconsistent theory, we see that the pair $\langle Y, \mathcal{L} \rangle$ is an M-algebra, because Y is closed under the measurements in $\mathcal{L}$. In this M-algebra, all measurements do more than commute, they are *classical*, in the following sense.

Definition 5 *A mapping* $\alpha : X \longrightarrow X$ *is said to be* classical *iff for every* $x \in X$, *either* $\alpha(x) = x$ *or* $\alpha(x) = 0$.

The M-algebra above is separable: if T_1 and T_2 are diferent maximal consistent theories there is a formula $a \in T_1 - T_2$. It is not strongly separable, though, if there is no single formula equivalent to a maximal theory.

4.1.2 Nonmonotonic inference operations

In Section 4.1.1 we assumed that the inference operation $\mathcal{C}n$ was monotonic. It seems attractive to consider the more general case of nonmonotonic inference operations studied, for example in [6]. More precisely what about replacing **Monotonicity** by the weaker

$$\textbf{Cumulativity}\ \forall A, B \subseteq \mathcal{L},\ A \subseteq B \subseteq \mathcal{C}(A) \ \Rightarrow\ \mathcal{C}(B) = \mathcal{C}(A).$$

Notice that, in such a case, we prefer to denote our inference operation by $\mathcal{C}$ and not by $\mathcal{C}n$. The reader may verify that all requirements for an M-algebra still hold true, *except for* **Composition**. In such a structure all measurements still commute and we therefore need that every composition $a \circ b$ of measurements (formula) be a measurement (formula). But the reader may check that $a \wedge b$ does not have the required properties: $\mathcal{C}(T, a \wedge b) = \mathcal{C}(T, a, b)$ but, since $\mathcal{C}$ is not required to be monotonic, there may well be some formula $c \in \mathcal{C}(T, a)$ that is not in $\mathcal{C}(T, a, b)$. In such a case $\mathcal{C}(T, a \wedge b) \neq \mathcal{C}(\mathcal{C}(T, a), b)$, as would be required. One may, then, think of extending the language $\mathcal{L}$ to include formulas of the form $a \circ b$ acting as compositions. But the **Negation** condition of the definition of an M-algebra requires every formula (measurement) to have a negation and there is no obvious definition for the negation of a composition. The **Monotonicity** property seems therefore essential.

4.1.3 Revisions

Another natural idea is to consider revisions a la AGM [1]. The action of a formula a on a theory T would be defined as the theory T revised by a: $T{*}a$. The structure obtained does not satisfy the M-algebra assumptions. The most blatant violation concerns **Negation**. In revision theory negation does not behave at all as expected in an M-algebra.

4.2 Orthomodular and Hilbert spaces

J. von Neumann's [12] firmly set Quantum Mechanics in the framework of Hilbert spaces. We assume the definition of a Hilbert space is known to the reader. Hilbert spaces are orthomodular spaces. We shall not burden the reader with the definition of such spaces here: the reader may replace, in the sequel, the word *orthomodular* by *Hilbert* and lose little of the strength of the results. The reader may want to consult [11]. A fundamental (but not used in this paper) result of Solèr [10] characterizes infinite-dimensional Hilbert spaces in orthomodular spaces.

4.2.1 Orthomodular spaces

Given any orthomodular space $\mathcal{H}$, denote by M the set of all closed subspaces of $\mathcal{H}$. Then the pair $\langle \mathcal{H}, M \rangle$ is an M-algebra, if any $\alpha \in M$ acts on $\mathcal{H}$ in the following way: $\alpha(x)$ is the unique vector such that $x = \alpha(x) + y$ for some vector $y \in \alpha^{\perp}$. In light of Section 3 the reader will have no trouble proving that any such structure is an M-algebra. It is not separable, though: any two colinear vectors satisfy exactly the same measurements. The next section will present a related separable M-algebra.

4.2.2 Rays

Given any orthomodular space $\mathcal{H}$, let X be the set of one-dimensional or zero-dimensional subspaces of $\mathcal{H}$. Let M be the set of closed subspaces of $\mathcal{H}$. The projection on a closed subspace is linear and therefore sends a one-dimensional subspace to a one-dimensional or a zero-dimensional subspace and sends the zero-dimensional subspace to itself. The pair $\langle X, M \rangle$ is easily seen to be an M-algebra. This M-algebra is strongly separable: notice that $X \subset M$ and that $x \in X$, $x \neq \vec{0}$ is the only non-null state satisfying the measurement x.

5 Properties of M-algebras

We assume that $\langle X, M\rangle$ is an arbitrary M-algebra. First, we shall show that any M-algebra includes two trivial measurements: $\top$, analogous to the truth-value *true*, that leaves every state unchanged and measures a property satisfied by every state and $\perp$, analogous to *false*, that sends every state to the illegitimate state, and is nowhere satisfied.

Lemma 1 *[Negation, Composition, Idempotence] There are measurements $\top, \perp \in M$ such that for every $x \in X$, $\top(x) = x$ and $\perp(x) = 0$.*

Proof: The set M of measurements is not empty: assume $\alpha \in M$. Clearly, by **Negation**, the measurement $\neg\alpha$ preserves α. It follows, by **Composition**, that $\alpha \circ (\neg\alpha)$ is a measurement. Let $\perp = \alpha \circ (\neg\alpha)$. By **Idempotence** and **Negation**, for every $x \in X$, $\perp(x) = 0$. We now let $\top = \neg\perp$. ▮

Then, we want to show that measurements are uniquely specified by their fixpoints.

Lemma 2 *[Idempotence, Cumulativity] For any $\alpha, \beta \in M$, if $FP(\alpha) = FP(\beta)$, then $\alpha = \beta$.*

Proof: Assume $FP(\alpha) = FP(\beta)$. Let $x \in X$. By **Idempotence** $\alpha(x) \in FP(\alpha)$ and therefore, by assumption $\alpha(x) \in FP(\beta)$. Similarly $\beta(x) \in FP(\alpha)$. By **Cumulativity**, then, $\alpha = \beta$. ▮

Corollary 1 *[Idempotence, Cumulativity, Negation] For any $\alpha \in M$, $\neg\neg\alpha = \alpha$.*

Proof: Both α and $\neg\neg\alpha$ are measurements and $FP(\neg\neg\alpha) = FP(\alpha)$. ▮

We shall now prove a very important property. Suppose x is a state in which some measurement (i.e., proposition) holds: for example, at x the spin along the x-axis is $1/2$. Performing a measurement α on x may lead to a different state $y = \alpha(x)$. At y the spin along the x-axis may still be $1/2$, or it may be the case that the measurement α has interfered with the value of the spin. But, under no circumstance, can it be the case that the spin along the x-axis has a definite value different from $1/2$, such as $-1/2$. If the value of the spin along the x-axis at y is not $1/2$, the spin must be indefinite. This expresses the fact that a measurement α, acting on a state in which β holds, can either preserve β (when $\alpha(x) \in FP(\beta)$) or can disturb β (when

$\alpha(x) \not\in Def(\beta)$) but cannot make β impossible at x, i.e., $\alpha(x) \in Z(\beta)$. This is a very natural requirement stemming from the *minimal change* principle. A move from a definite value to a different definite value is too drastic to be accepted as measurement.

In the Hilbertian presentation of Quantum Physics, measurements are projections. The projection of a non-null vector x onto a closed subspace A is never orthogonal to x, unless x is orthogonal to A. Therefore if x is in some subspace B, but its projection on A is orthogonal to B, then this projection is the null vector.

Lemma 3 *[Illegitimate, Interference] For any $x \in X$, $\alpha, \beta \in M$, if $x \in FP(\beta)$, i.e., $\beta(x) = x$, and $\alpha(x) \in Z(\beta)$, i.e., $\beta(\alpha(x)) = 0$, then $x \in Z(\alpha)$, i.e., $\alpha(x) = 0$.*

Proof: Assume $x \in FP(\beta)$ and $\beta(\alpha(x)) = 0$. Then $(\alpha \circ \beta)(x) = 0 \in FP(\alpha)$. By **Interference**, then, $\alpha(x) \in FP(\beta)$ and $\beta(\alpha(x)) = \alpha(x)$, i.e., $0 = \alpha(x)$. ∎

We shall now sort out the relation between fixpoints and zeros. The next result is a dual of Lemma 3.

Lemma 4 *[Illegitimate, Interference, Negation] $\forall x \in X$, $\forall \alpha, \beta \in M$, if $x \in Z(\beta)$ and $\alpha(x) \in FP(\beta)$, then $x \in Z(\alpha)$. In other terms, if $\beta(x) = 0$ and $\beta(\alpha(x)) = \alpha(x)$, then $\alpha(x) = 0$.*

Proof: Consider the measurement $\neg\beta$ guaranteed by **Negation**. If we have $x \in FP(\neg\beta)$ and $\alpha(x) \in Z(\neg\beta)$, then, by Lemma 3 we have $x \in Z(\alpha)$. ∎

Lemma 5 *[Illegitimate, Idempotence, Interference, Negation] For any $\alpha, \beta \in M$, $FP(\alpha) \subseteq FP(\beta)$ iff $Z(\beta) \subseteq Z(\alpha)$.*

Proof: Suppose $FP(\alpha) \subseteq FP(\beta)$ and $x \in Z(\beta)$. Since, by **Idempotence**, $\alpha(x) \in FP(\alpha)$, we have, by assumption, $\alpha(x) \in FP(\beta)$. By Lemma 4, then $x \in Z(\alpha)$.

Suppose now that $Z(\beta) \subseteq Z(\alpha)$. We have $FP(\neg\beta) \subseteq FP(\neg\alpha)$ and by what we just proved: $Z(\neg\alpha) \subseteq Z(\neg\beta)$. We conclude that $FP(\alpha) \subseteq FP(\beta)$. ∎

We shall now consider the composition of measurements. First we show the symmetry of the preservation relation.

Lemma 6 *[Idempotence, Interference] For any $\alpha, \beta \in M$, α preserves β iff β preserves α.*

Proof: Assume α preserves β, and $x \in FP(\alpha)$. By **Idempotence**, $\beta(x) \in FP(\beta)$. Since α preserves β, $\alpha(\beta(x)) \in FP(\beta)$. The assumptions of **Interference** are satisfied and we conclude that $\beta(x) \in FP(\alpha)$. We have shown that β preserves α. ▮

Lemma 7 *[Illegitimate, Idempotence, Interference, Negation] For any $\alpha, \beta \in M$, if $\alpha \circ \beta \in M$, then $FP(\alpha \circ \beta) = FP(\alpha) \cap FP(\beta)$.*

Proof: Since $Z(\alpha) \subseteq Z(\alpha \circ \beta)$, Lemma 5 implies that $FP(\alpha \circ \beta) \subseteq FP(\alpha)$. By **Idempotence** of β, $FP(\alpha \circ \beta) \subseteq FP(\beta)$. We see that $FP(\alpha \circ \beta) \subseteq FP(\alpha) \cap FP(\beta)$. But the inclusion in the other direction is obvious. ▮

We shall now show that the converse of **Composition** holds.

Lemma 8 *[Illegitimate, Idempotence, Interference, Negation] For any $\alpha, \beta \in M$, if $\alpha \circ \beta \in M$, then β preserves α.*

Proof: By Lemma 7, $FP(\alpha \circ \beta) \subseteq FP(\alpha)$. For any x, $(\alpha \circ \beta)(x)$ is therefore a fixpoint of α. Assume $x \in FP(\alpha)$. Then, $(\alpha \circ \beta)(x) = \beta(x)$ is a fixpoint of α. ▮

Lemma 9 *[Illegitimate, Idempotence, Interference, Composition, Negation] For any $\alpha, \beta \in M$, $\alpha \circ \beta \in M$, iff β preserves α.*

Proof: The *only if* part is Lemma 8. The *if* part is **Composition**. ▮

Lemma 10 *[Illegitimate, Idempotence, Interference, Composition, Negation] For any $\alpha, \beta \in M$, $\alpha \circ \beta \in M$ iff $\beta \circ \alpha \in M$.*

Proof: By Lemmas 9 and 6. ▮

Lemma 11 *[Illegitimate, Idempotence, Interference, Composition, Cumulativity, Negation] For any $\alpha, \beta \in M$, $\alpha \circ \beta \in M$ iff α and β commute, i.e., $\alpha \circ \beta = \beta \circ \alpha$.*

Proof: Assume, first, that $\alpha \circ \beta \in M$. By Lemma 10, $\beta \circ \alpha \in M$. By Lemma 7, $FP(\alpha \circ \beta) = FP(\beta \circ \alpha)$, which implies the claim by Lemma 2.

Assume, now that α and β commute. We claim that α preserves β: indeed, if $\beta(x) = x$, then $\beta(\alpha(x)) = \alpha(\beta(x)) = \alpha(x)$ and therefore, by **Composition**, $\beta \circ \alpha$ is a measurement. ▮

Lemma 12 *[Illegitimate, Idempotence, Interference, Composition, Cumulativity, Negation] For any* $\alpha, \beta \in M$*, if* $FP(\alpha) \subseteq FP(\beta)$*, then* $\alpha \circ \beta = \beta \circ \alpha = \alpha$*.*

Proof: If $FP(\alpha) \subseteq FP(\beta)$, then, clearly $\alpha \circ \beta = \alpha$ by Idempotence of α. Therefore $\alpha \circ \beta \in M$ and, by Lemma 11, α and β commute. ∎

6 Connectives in M-algebras

6.1 Connectives for arbitrary measurements

The reader has noticed that *negation* plays a central role in our presentation of M-algebras, through the **Negation** requirement and that this requirement is central in the derivation of many of the lemmas of Section 5. Indeed, **Negation** expresses the orthogonality structure so fundamental in orthomodular and Hilbert spaces. The requirement of **Negation** corresponds, for the logician, to the existence of a connective whose properties are those of a classical negation. Indeed, for example, as shown by Corollary 1, double negations may be ignored, as is the case in classical logic. In [2], the logical language presented includes negation, interpreted as orthogonal complement, and this is consistent with our interpretation. But [2] also defines other connectives: conjunction, disjunction and many later works on Quantum Logic define also implication (sometimes a number of implications). Our treatment does not require such connectives, or more precisely, our treatment does not require that such connectives be defined between any pair of measurements.

Consider conjunction. One may consider only M-algebras in which, for any $\alpha, \beta \in M$ there is a measurement $\alpha \wedge \beta \in M$ such that $FP(\alpha \wedge \beta) = FP(\alpha) \cap FP(\beta)$. There are many such M-algebras, since any M-algebra defined by an orthomodular space and the family of all its (projections on) closed subspaces has this property since the intersection of any two closed subspaces is a closed subspace. But our requirements do not imply the existence of such a measurement $\alpha, \beta \in M$ for every α and β.

For disjunction, one may consider requiring that for any $\alpha, \beta \in M$ there be a measurement $\alpha \vee \beta \in M$ such that $Z(\alpha \vee \beta) = Z(\alpha) \cup Z(\beta)$, and the M-algebras defined by Hilbert spaces satisfy this requirement. Not all M-algebras satisfy this requirement.

For implication, in general M-algebras, assuming conjunction and disjunction, one could require that for any $\alpha, \beta \in M$ there be a measurement $\alpha \rightarrow \beta \in M$ such that $FP(\alpha \rightarrow \beta) = FP(\neg\alpha \vee (\alpha \wedge \beta))$, and the M-algebras defined by Hilbert spaces satisfy this requirement. Indeed, works in Quantum Logic sometimes consider more than one implication, see [3].

The thesis of this paper is that connectives should not be defined for arbitrary measurements, but only for commuting measurements. One of the novel features of M-algebras is that conjunction, disjunction and implication are defined only for commuting measurements. The next section will show that this restriction leads to a classical propositional logic. If one restricts oneself to commuting measurements, then, contrary to the unrestricted connectives of Birkhoff and von Neumann [2], conjunction and disjunction distribute, and, in fact, the logic obtained is classical.

6.2 Connectives for commuting measurements

Let us take a second look at propositional connectives in M-algebras, with particular attention to their commutation properties. We shall assume that $\langle X, M \rangle$ is an M-algebra.

6.2.1 Negation

Negation asserts the existence of a negation for every measurement. Let us study the commutation properties of $\neg\alpha$.

Lemma 13 *$\forall \alpha, \beta \in M$, if α commutes with β, then $\neg\alpha$ commutes with β.*

Proof: Assume α commutes with β. We shall see that β preserves $\neg\alpha$. Let $x \in FP(\neg\alpha)$. We have $x \in Z(\alpha)$. But $(\alpha \circ \beta)(x) = (\beta \circ \alpha)(x)$. Therefore $0 = \alpha(\beta(x))$, $\beta(x) \in Z(\alpha)$ and $\beta(x) \in FP(\neg\alpha)$. We have shown that β preserves $\neg\alpha$. By **Composition**, $(\neg\alpha) \circ \beta \in M$ and, by Lemma 11, $\neg\alpha$ commutes with β. ∎

Corollary 2 *$\forall \alpha, \beta \in M$, α and β commute iff $\neg\alpha$ and β commute iff α and $\neg\beta$ commute iff $\neg\alpha$ and $\neg\beta$ commute.*

Proof: By Lemma 13 and Corollary 1. ∎

6.2.2 Conjunction

We shall now define a conjunction between *commuting* measurements.

Definition 6 *For any* commuting *measurements* $\alpha, \beta \in M$*, the conjunction* $\alpha \wedge \beta$ *is defined by:* $\alpha \wedge \beta = \alpha \circ \beta = \beta \circ \alpha$.

By Lemma 11, the conjunction, as defined, is indeed a measurement.

Lemma 14 *For any commuting* $\alpha, \beta \in M$*, the conjunction* $\alpha \wedge \beta$ *is the unique measurement* γ *such that* $FP(\gamma) = FP(\alpha) \cap FP(\beta)$.

Proof: By Lemmas 2 and 7. ▮

One immediately sees that conjunction among commuting measurements is associative, commutative and that $\alpha \wedge \alpha = \alpha$ for any $\alpha \in M$.

Let us now study the commutation properties of conjunction.

Lemma 15 $\forall \alpha, \beta, \gamma \in M$*, that commute in pairs,* $\alpha \wedge \beta$ *commutes with* γ.

Proof:

$$(\alpha \wedge \beta) \circ \gamma = (\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) = \alpha \circ (\gamma \circ \beta) =$$

$$(\alpha \circ \gamma) \circ \beta = (\gamma \circ \alpha) \circ \beta = \gamma \circ (\alpha \circ \beta) = \gamma \circ (\alpha \wedge \beta)$$

▮

6.2.3 Disjunction

One may now define a disjunction between two commuting measurements in the usual, classical, way.

Definition 7 *For any* commuting *measurements* $\alpha, \beta \in M$*, the disjunction* $\alpha \vee \beta$ *is defined by:* $\alpha \vee \beta = \neg(\neg\alpha \wedge \neg\beta)$.

By Corollary 2, the measurements $\neg\alpha$ and $\neg\beta$ commute, therefore their conjunction is well-defined and the definition of disjunction is well-formed.

The commutation properties of disjunction are easily studied.

Lemma 16 $\forall \alpha, \beta, \gamma \in M$ *that commute in pairs,* $\alpha \vee \beta$ *commutes with* γ.

Proof: Obvious from Definition 7 and Lemmas 13 and 15. ▮

The following is easily proved: use Definition 7, **Negation** and Lemmas 5, 2 and 11.

Lemma 17 *For any commuting measurements, α and β, their disjunction $\alpha \vee \beta$ is the unique measurement γ such that $Z(\gamma) = Z(\alpha) \cap Z(\beta)$.*

Lemma 18 *If $\alpha, \beta \in M$* commute, *then $FP(\alpha) \cup FP(\beta) \subseteq FP(\alpha \vee \beta)$.*

The inclusion is, in general, strict.
Proof: Since $Z(\alpha \vee \beta) \subseteq Z(\alpha)$, by Lemma 5. ∎

Contrary to what holds in classical logic, in M-algebras we can have a state x that satisfies the disjunction $\alpha \vee \beta$ but does not satisfy any one of α or β. This is particularly interesting when α and β represent measurements of different values for the same physical quantity. In this case, one is tempted to say that such an x satisfies α not entirely but in *part* and β in some other part. In the Hilbertian formalism x is a linear combination of the two vectors $\alpha(x)$ and $\beta(x)$: $x = c_1\alpha(x) + c_2\beta(x)$. The coefficients c_1 and c_2 describe in what proportions the state x, that satisfies $\alpha \vee \beta$ satisfies α and β respectively. The consideration of structures richer than M-algebras that include this quantitative information is left for future work.

6.2.4 Implication

Implication ($\rightarrow$) is probably the most interesting connective. It will play a central role in our treatment of connectives.

Definition 8 *For any* commuting *measurements $\alpha, \beta \in M$, the implication $\alpha \rightarrow \beta$ is defined by: $\alpha \rightarrow \beta = \neg(\alpha \wedge \neg\beta)$.*

By Corollary 2, the measurements α and $\neg\beta$ commute, therefore their conjunction is well-defined and the definition of implication is well-formed.

The commutation properties of implication are easily studied.

Lemma 19 *$\forall \alpha, \beta, \gamma \in M$ that commute in pairs, $\alpha \rightarrow \beta$ commutes with γ.*

Proof: Obvious from Definition 8 and Lemmas 13 and 15. ∎

The following is easily proved: use Definition 8, **Negation** and Lemmas 5, 2 and 11.

Lemma 20 *For any commuting measurements, α and β, their implication $\alpha \rightarrow \beta$ is the unique measurement γ such that $Z(\gamma) = FP(\alpha) \cap Z(\beta)$.*

Lemma 20 characterizes the zeros of $\alpha \rightarrow \beta$. Our next result characterizes the fixpoints of $\alpha \rightarrow \beta$ in a most telling and useful way.

Lemma 21 *For any commuting measurements, α and β, their implication $\alpha \to \beta$ is the unique measurement γ such that $FP(\gamma) = \{x \in X \mid \alpha(x) \in FP(\beta)\}$.*

Proof: Assume α and β commute, and $x \in X$. Now, $\alpha(x) \in FP(\beta)$ iff (by **Negation**) $\alpha(x) \in Z(\neg\beta)$ iff $(\alpha \circ (\neg\beta))(x) = 0$ iff (by Definition 6 and Lemma 13) $(\alpha \wedge (\neg\beta))(x) = 0$ iff $x \in Z(\alpha \wedge (\neg\beta))$ iff (by **Negation**) $x \in FP(\neg(\alpha \wedge (\neg\beta)))$ iff (by Definition 8) $x \in FP(\alpha \to \beta)$. ∎

The following is immediate.

Corollary 3 *For any commuting measurements α and β, if $x \in FP(\alpha)$ and $x \in FP(\alpha \to \beta)$, then $x \in FP(\beta)$.*

One may now ask whether the propositional connectives we have defined amongst commuting measurements behave classically. In particular, assuming that measurements α, β and γ commute in pairs, does the distribution law hold, i.e., is it true that $(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$. In the next section, we shall show that amongst commuting measurements propositional connectives behave classically.

7 Amongst commuting measurements connectives are classical

Let us, first, remark on the commutation properties described in Lemmas 13, 15, 16 and 19. Those lemmas imply that, given any set $A \subseteq M$ of measurements in an M-algebra, such that any two elements of A commute, one may consider the propositional calculus built on A (as atomic propositions). Each such proposition describes a measurement in the original M-algebra (an element of M) and all such measurements commute. We shall denote by $Prop(A)$ the propositions built on A.

We shall now show that, in any such $Prop(A)$ all classical propositional tautologies hold at every state $x \in X$.

Theorem 1 *Let $\langle X, M \rangle$ be an M-algebra. Let $A \subseteq M$ be a set of pairwise* commuting *measurements. If $\alpha \in Prop(A)$ is a classical propositional tautology, then $FP(\alpha) = X$.*

The converse does not hold, since it is easy to build M-algebras in which, for example, a given measurement holds at every state.

We shall use the axiomatic system for propositional calculus found on p. 31 of Mendelson's [8] to prove that any classical tautology α built out using only negation and implication has the property claimed. We shall then show that conjunction and disjunction may be defined in terms of negation and implication as usual. The proof will proceed in six steps: Modus Ponens, the three axiom schemes of Mendelson's system, conjunction and disjunction. The reader should notice how tightly the three axiom schemas correspond to the commutation assumption.

Lemma 22 *For any commuting measurements α and β, if $FP(\alpha) = X$ and $FP(\alpha \rightarrow \beta) = X$, then $FP(\beta) = X$.*

Proof: By Corollary 3. ▮

Lemma 23 *For any commuting measurements α and β,*

$$FP(\alpha \rightarrow (\beta \rightarrow \alpha)) = X.$$

Proof: Since α and β commute, for any $x \in X$: $\beta(\alpha(x)) = \alpha(\beta(x))$, therefore, by **Idempotence**, we have $\beta(\alpha(x)) \in FP(\alpha)$. By Lemma 21, for any x, $\alpha(x) \in FP(\beta \rightarrow \alpha)$. By the same lemma: $x \in FP(\alpha \rightarrow (\beta \rightarrow \alpha))$. ▮

Lemma 24 *For any pairwise commuting measurements α, β and γ*

$$FP((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))) = X.$$

Proof: By Lemma 21, it is enough to show that for any $x \in X$, if $y = (\alpha \rightarrow (\beta \rightarrow \gamma))(x)$, then, if we define $z = (\alpha \rightarrow \beta)(y)$, and define $w = \alpha(z)$, then we have: $\gamma(w) = w$. But since all the measurements above commute, by **Idempotence**, the state w satisfies $\alpha \rightarrow (\beta \rightarrow \gamma)$, $\alpha \rightarrow \beta$ and α. By Corollary 3, w satisfies β and $\beta \rightarrow \gamma$. For the same reason w satisfies γ. ▮

Lemma 25 *For any commuting measurements α and β,*

$$FP((\neg\beta \rightarrow \neg\alpha) \rightarrow ((\neg\beta \rightarrow \alpha) \rightarrow \beta)) = X.$$

Proof: By Lemma 21, it is enough to show that for any $x \in X$, if $y = (\neg\beta \rightarrow \neg\alpha)(x)$, then, if we define $z = (\neg\beta \rightarrow \alpha)(y)$ then we have: $\beta(z) = z$. But since all the measurements above commute, by **Idempotence**, the state z satisfies $\neg\beta \rightarrow \neg\alpha$ and $\neg\beta \rightarrow \alpha$. Therefore, by Lemma 21, $(\neg\beta)(z)$ satisfies both $\neg\alpha$ and α. Therefore $(\neg\beta)(z) = 0$ and therefore, by **Negation**, $z \in FP(\beta)$. ▮

Lemma 26 *For any commuting measurements α and β, $\alpha \wedge \beta = \neg(\alpha \rightarrow \neg\beta)$.*

Proof:

$$FP(\neg(\alpha \rightarrow \neg\beta)) = Z(\alpha \rightarrow \neg\beta) = FP(\alpha) \cap Z(\neg\beta) = FP(\alpha) \cap FP(\beta).$$

By **Negation**, Lemma 20 and **Negation**. The conclusion then follows from Lemma 14. ▮

Lemma 27 *For any commuting measurements α and β, $\alpha \vee \beta = (\neg\alpha) \rightarrow \beta$.*

Proof:

$$Z((\neg\alpha) \rightarrow \beta) = FP(\neg\alpha) \cap Z(\beta) = Z(\alpha) \cap Z(\beta).$$

By Lemma 20 and **Negation**. The conclusion then follows from Lemma 17. ▮

We have proved Theorem 1.

The following is a Corollary.

Corollary 4 *Let $\langle X, M\rangle$ be an M-algebra. Let $A \subseteq M$ be a set of pairwise* commuting *measurements. If $\alpha, \beta \in Prop(A)$ are such that α logically implies β, i.e., $\alpha \models \beta$, then $FP(\alpha) \subseteq FP(\beta)$. If α and β are logically equivalent, they are equal.*

Proof: Since $\alpha \rightarrow \beta$ is a tautology, by Theorem 1, $FP(\alpha \rightarrow \beta) = X$. By Lemma 21, then, $FP(\alpha) \subseteq FP(\beta)$. If α and β are logically equivalent, they have the same set of fixpoints by the above, and, by Lemma 2 they are equal. ▮

8 Orthomodularity of M-algebras

In this section we shall clear up the relation between our M-algebras and the lattice structures largely studied previously (see [9] for an in-depth study and review). The set of measurements M of an M-algebra is naturally endowed with a partial order.

Definition 9 *Let $\langle X, M\rangle$ be an M-algebra. For any $\alpha, \beta \in M$, let $\alpha \leq \beta$ iff $FP(\alpha) \subseteq FP(\beta)$ (or equivalently, by Lemma 5, iff $Z(\beta) \subseteq Z(\alpha)$).*

Lemma 28 *Let $\langle X, M\rangle$ be an M-algebra. The relation $\leq$ on M is a partial order. The measurement $\top$ is the top element and the measurement $\perp$ is the bottom element, i.e.: for any $\alpha \in M$, $\perp \leq \alpha \leq \top$. Any two* commuting *measurements have a greatest lower bound and a least upper bound in M.*

The set M is not, in general, a lattice, under $\leq$.
Proof: The relation $\leq$ is a partial order because $\subseteq$ is. For any $\alpha \in M$, $FP(\perp) = \{0\} \subseteq FP(\alpha) \subseteq X = FP(\top)$. Consider any two commuting measurements α and β. By Lemma 14 the measurement $\alpha \wedge \beta$ is the greatest lower bound of α and β. By Lemma 17 and the definition of $\leq$ in terms of Z it is clear that $\alpha \vee \beta$ is the least upper bound of α and β. ▮

M-algebras represent a departure from the structures previously considered by researchers in Quantum Logic because they are not lattices. Orthomodular lattices have been considered by most to be the structure of choice. Orthomodular lattices are lattices equipped with a unary operation of orthocomplementation $^\perp$ satisfying the following properties (see for example [9], pages 33-35). For any α, β, γ:

1. $(\alpha^\perp)^\perp = \alpha$,
2. if $\alpha \leq \beta$, then $\beta^\perp \leq \alpha^\perp$,
3. the greatest lower bound of α and $\alpha^\perp$ is the bottom element,
4. the least upper bound of α and $\alpha^\perp$ is the top element, and
5. if $\alpha \leq \beta$, then $\alpha \vee (\neg\alpha \wedge \beta) = \beta$.

Lemma 29 *Let $\langle X, M\rangle$ be an M-algebra. Each one of the properties above holds for the partially ordered set $\langle M, \leq\rangle$ when negation is taken for orthocomplementation.*

Proof: Item 1 holds by Corollary 1. Item 2 holds by **Negation** and Lemma 5. Item 3 holds since α and $\neg\alpha$ commute, by Lemma 28, their greatest lower bound is $\alpha \wedge \neg\alpha$ and $FP(\alpha \wedge \neg\alpha) = FP(\alpha) \cap Z(\alpha) = \{0\} = FP(\perp)$. Item 4 holds similarly: $Z(\alpha \vee \neg\alpha) = \{0\} = Z(\top)$. Item 5 follows from the following considerations. If $\alpha \leq \beta$, then α and β commute by Lemma 12. Therefore, by Lemma 13, $\neg\alpha$ and β commute, and by Lemma 28 $\gamma = \neg\alpha \wedge \beta$. Also, α and γ commute by Lemma 16 and by Lemma 28 all we have to show is that $\beta = \alpha \vee (\neg\alpha \wedge \beta)$. But β is logically equivalent to $(\alpha \wedge \beta) \vee (\neg\alpha \wedge \beta)$.

Corollary 4 implies that $\beta = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \beta)$. By assumption $\alpha \leq \beta$ and, by Lemma 12 $\alpha \wedge \beta = \alpha$. We conclude that $\beta = \alpha \vee (\neg\alpha \wedge \beta)$. ∎

The next Section will consider separable and strongly separable M-algebras.

9 Separable and stronly separable M-algebras

9.1 Separable M-algebras

Lemma 30 *In a* separable *M-algebra, a measurement is classical if and only if it commutes with any measurement.*

Proof: Suppose α is classical. Consider any $x \in X$ and any $\beta \in M$. Since α is classical we know that $x \in FP(\alpha)$ or $x \in Z(\alpha)$ and $\beta(x) \in FP(\alpha)$ or $\beta(x) \in Z(\alpha)$. If $x \in FP(\alpha)$, by Lemma 3, $\beta(x) \in Z(\alpha)$ implies $x \in Z(\beta)$and $(\alpha \circ \beta)(x) = 0 = (\beta \circ \alpha)(x)$. But $\beta(x) \in FP(\alpha)$ implies $(\alpha \circ \beta)(x) = \beta(x) = (\beta \circ \alpha)(x)$.

If $x \in Z(\alpha)$ and $\beta(x) \in FP(\alpha)$, by Lemma 4, $\beta(x) = 0$ and $(\alpha \circ \beta)(x) = 0 = (\beta \circ \alpha)(x)$. If $\beta(x) \in Z(\alpha)$, then $(\alpha \circ \beta)(x) = 0 = (\beta \circ \alpha)(x)$.

Suppose, now that α commutes with any measurement β. By contradiction, assume $\alpha(x) \neq 0$ and $\alpha(x) \neq x$. By **Separability** there is some measurement γ such that $x \in FP(\gamma)$ and $\alpha(x) \notin FP(\gamma)$. But α and γ commute and: $(\alpha \circ \gamma)(x) = (\gamma \circ \alpha)(x) = \alpha(x)$. We see that $\alpha(x) \in FP(\gamma)$, a contradiction. ∎

Note that a measurement α is classical (see Definition 5) iff $Def(\alpha) = X$.

Lemma 31 *If α is classical, so is $\neg\alpha$. If α and β are classical, then so are $\alpha \wedge \beta$, $\alpha \vee \beta$ and $\alpha \rightarrow \beta$.*

Proof: If α is classical, $Def(\alpha) = X$ and therefore $Def(\neg\alpha) = X$. For conjunction $(\alpha \wedge \beta)(x) = (\alpha \circ \beta)(x) = (\beta \circ \alpha)(x)$. If either $\alpha(x)$ or $\beta(x)$ is 0 then $(\alpha \wedge \beta)(x) = 0$, otherwise $\alpha(x) = x = \beta(x)$ and $(\alpha \wedge \beta)(x) = x$. The definitions of disjunction and implication in terms of negation and conjunction, then ensure the claim. ∎

9.2 Strongly separable M-algebras

We shall show that much of the linear dependency structure of orthomodular and Hilbert spaces is present in any strongly separable M-algebra. Theorem 2 shows that the action of the measurements in a strongly separable M-algebra is already encoded in its order structure.

Theorem 2 *Let $\langle X, M\rangle$ be any strongly separable M-algebra. The following two properties are satisfied for any $\alpha \in M$ and any $x \in X$:*

1. $x \in FP(\alpha \rightarrow e_{\alpha(x)})$, *and*
2. *if $x \notin Z(\alpha)$, there exists a unique $y \in FP(\alpha)$ such that $x \in FP(\alpha \rightarrow e_y)$. This y is $\alpha(x)$.*

Proof: Note, first that the measurements $\alpha \rightarrow e_{\alpha(x)}$ and $\alpha \rightarrow e_y$ are well-defined since α and $e_{\alpha(x)}$ commute by Lemma 12 since $FP(e_{\alpha(x)}) = \{0, x\} \subseteq FP(\alpha)$ by Idempotence, and, similarly α and e_y commute if $y \in FP(\alpha)$.

Both claims follow straightforwardly from Lemma 21 and the following: $\alpha(x) \in FP(e_{\alpha(x)})$, and $\alpha(x) \in FP(e_y)$ iff $\alpha(x) = 0$ or $\alpha(x) = y$. ∎

The following shows the decomposition of any state in its orthogonal components.

Corollary 5 *Let $\langle X, M\rangle$ be any strongly separable M-algebra. For any $\alpha \in M$ and any $x \in X$, $x \in FP(e_{\alpha(x)} \vee e_{(\neg\alpha)(x)})$.*

Proof: Note that, since $e_{\alpha(x)}$ and $e_{(\neg\alpha)(x)}$ are orthogonal, they commute and the disjunction of the claim is well defined.

By Theorem 2, we have both $x \in FP(\neg\alpha \vee e_{\alpha(x)})$ and $x \in FP(\alpha \vee e_{(\neg\alpha)(x)})$. But all measurements mentioned above commute and the conjunction of the two disjunctions is well-defined and $x \in FP((\alpha \vee e_{(\neg\alpha)(x)}) \wedge (\neg\alpha \vee e_{\alpha(x)}))$. But, by Theorem 1:

$$(\alpha \vee e_{(\neg\alpha)(x)}) \wedge (\neg\alpha \vee e_{\alpha(x)}) = e_{(\neg\alpha)(x)} \vee e_{\alpha(x)}.$$

∎

10 Reflections and further work

The formalism of M-algebras is weaker than that of Hilbert spaces and may be motivated by epistemological concerns. Nevertheless some of the properties of quantum measurements may be understood in this weaker formalism. Should additional principles be incorporated into M-algebras? Probably, the next step will be to incorporate some quantitative information about the relation between a state and a measurement.

Corollary 30 may explain why elementary particles all have a definite value for their total spin. The corresponding Hermitian operator commutes with every spin operator, and most probably any physically meaningful operator on an isolated particle. Therefore the corresponding measurements (of the different values of the total spin) are classical. This means that, for any elementary particle its total spin has a definite value. Particles with different definite values for their total spin are better considered different particles: no measurement can change their total spin.

Can the formalism of M-algebras throw light on superselection rules, such as the symmetrization postulate?

Another tantalizing question concerns the tensor product of M-algebras. What should it be? Would this explain why a quantic system composed of two separate subsystems must be represented by the tensor product of the spaces representing the subsystems?

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